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An Exponential Upper Bound for the Survival Probability in a Dynamic Random Trap Model

F. Redig[1]

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We consider a symmetric translation-invariant random walk on the d-dimensional lattice $\mathbb{Z}^d$. The walker moves in an environment of moving traps. When the walker hits a trap, he is killed. The configuration of traps in the course of time is a reversible Markov process satisfying a level-2 large-deviation principle. Under some restrictions on the entropy function, we prove an exponential upper bound for the survival probability, i.e.,

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(T \geqslant t) < 0$$

where T is the survival time of the walker. As an example, our results apply to a random walk in an environment of traps that perform a simple symmetric exclusion process.

KEY WORDS: Dynamic trap model; level-2 large-deviation principle; reversible Markov process; environment process; killing function; range of random walk.

1. INTRODUCTION

The trapping problem is one in which a particle moves about randomly in a space containing randomly located traps which may or may not themselves be mobile (see ref. 1 for a review on the different models and applications of trapping). The quantity of interest is the survival time T. More precisely, one wants to study the asymptotics for $t \to \infty$ of the probability $P(T \geqslant t)$ that the particle remains untrapped at least during the time interval $[0, t)$.

[1] Katholieke gemeenschap "Ster van David" Bredabaan 737, 2990 Gooreind, Belgium.

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In the case of simple random walk in a static Bernoulli distribution of traps, this problem reduces to the study of the large-deviation properties of the range R_n (i.e., the number of distinct sites visited by the walker after n steps). The result is[4]

$$\log P(T \geqslant t) \sim -\lambda \cdot t^{d/(d+2)} \qquad (t \to \infty) \tag{1.1}$$

where $d \geqslant 1$ is the dimension and λ is a constant which depends on the dimension and the density of traps.

When we turn to the case of moving traps, much less is known. In general it is believed that $P(T \geqslant t)$ decays exponentially fast (ref. 1, Section 6b), but as far as we know, no rigorous results have been obtained.

In the present paper we consider a particle in an environment of moving traps. The process of the trap configuration as function of time is assumed to be a reversible Markov process that satisfies a level-2 large-deviation principle[5,3] with some extra technical conditions on the entropy function. The random walk of the particle and the process of moving traps are independent processes. We prove an exponential upper bound for the asymptotics of $P(T \geqslant t)$. More precisely, we prove that

$$\limsup_{t \to \infty} \frac{1}{t} \log P(T \geqslant t) < 0 \tag{1.2}$$

As an example our result applies to a simple random walk on $\mathbb{Z}^d$ in an environment of traps that perform a simple symmetric exclusion process starting from a Bernoulli measure.

The paper is organized as follows. In Sections 2 and 3 we introduce the process of moving traps, and the random walk to be trapped. In Section 4 we introduce the *environment process*, i.e., the process of trap configurations as seen from the position of the random walker. We calculate its generator and study its reversible and ergodic measures. In Section 5 we define the trapping problem and introduce the survival time. In Section 6 we discuss the large-deviation conditions that the process of moving traps has to satisfy in order to get (1.2). In Section 7 we prove (1.2), and in Section 8 we discuss some examples.

2. THE PROCESS OF MOVING TRAPS

Let E be a Polish space and $\mathscr{E}$ its Borel σ-field. An element $\eta \in E$ is called a *configuration of traps*. We denote by $H(E, E)$ the group of homeomorphisms of E into itself, i.e.,

$$H(E, E) = \{\varphi: E \to E \mid \varphi \text{ is a continuous bijection}\} \tag{2.1}$$

We suppose that there exists a *group homomorphism*

$$\tau: \quad \mathbb{Z}^d \to H(E, E) \tag{2.2}$$

For $\eta \in E$, $x \in \mathbb{Z}^d$, the element $\tau_x \eta = \tau(x)(\eta) \in E$ is interpreted as the configuration η shifted by x, or the configuration η *as seen from position* x. For a continuous function $f: E \to \mathbb{R}$ we define $\tau_x f: E \to \mathbb{R}$ by $\tau_x f(\eta) = f(\tau_x \eta)$, and for a probability measure $\mu \in \mathscr{P}(E)$, we define $\tau_x \mu$ by

$$\int_E \tau_x \mu(d\eta)\, f(\eta) = \int_E \mu(d\eta)\, \tau_x f(\eta) \tag{2.3}$$

The process of moving traps is assumed to be a Feller process on E denoted by $\{\eta_t: t \geqslant 0\}$. The path space measure of this process starting from $\eta \in E$ is denoted by $\mathbb{P}^T_\eta$. This is a measure on the space of cadlag trajectories on E,

$$\Omega^T = D([0, \infty), E) \tag{2.4}$$

The Borel σ-field on Ω^T is denoted by $\mathscr{A}^T$. The generator of the Feller process $\{\eta_t: t \geqslant 0\}$ is denoted by L_0, i.e.,

$$L_0 f(\eta) = \lim_{t \downarrow 0} \frac{\mathbb{E}^T_\eta f(\eta_t) - f(\eta)}{t} \tag{2.5}$$

Assumption 1. The domain $D(L_0)$ of the generator is translation invariant, i.e., for all $f \in D(L_0)$, $\tau_x f \in D(L_0)$ for all $x \in \mathbb{Z}^d$. Next the process $\{\eta_t: t \geqslant 0\}$ has a *reversible and ergodic* translation-invariant measure $\mu_{\text{eq}} \in \mathscr{P}(E)$, i.e.:

(1) L_0 is self-adjoint on $L^2(\mu_{\text{eq}})$.

(2) μ_{eq} is ergodic for the process $\{\eta_t: t \geqslant 0\}$.

(3) $\tau_x \mu_{\text{eq}} = \mu_{\text{eq}}$, $\forall x \in \mathbb{Z}^d$.

Inner product in $L^2(\mu_{\text{eq}})$ is denoted by $(\cdot, \cdot)$.

When starting from μ_{eq} the process of moving traps is reversible and ergodic. Its path space measure is denoted by

$$\mathbb{P}^T_{\mu_{\text{eq}}} = \int \mu_{\text{eq}}(d\eta)\, \mathbb{P}^T_\eta \tag{2.6}$$

3. THE RANDOM WALK

The random walk is a Markov process $\{X_t: t \geqslant 0\}$ on $\mathbb{Z}^d$ with generator

$$L_{\text{RW}} f(x) = \sum_{y \in \mathbb{Z}^d} p(x, y)[f(y) - f(x)] \tag{3.1}$$

Assumption 2. $p(x, y)$ is a symmetric translation-invariant transition probability on $\mathbb{Z}^d$, i.e.,

$$p(x, y) = p(y, x) = p(0, y-x) \geqslant 0$$
$$\sum_{y \in \mathbb{Z}^d} p(x, y) = 1 \tag{3.2}$$

Let $\mathbb{P}_x^{\mathrm{RW}}$ be the path space measure of the process $\{X_t : t \geqslant 0\}$ starting at x. This is a measure on the space of cadlag trajectories on $\mathbb{Z}^d$ denoted by $\Omega^{\mathrm{RW}} = D([0, \infty), \mathbb{Z}^d)$. The transition probability at time t is denoted by $p_t(0, x)$, i.e.,

$$p_t(0, x) = \mathbb{P}_0^{\mathrm{RW}}(X_t = x) \tag{3.3}$$

The random walk $\{X_t : t \geqslant 0\}$ starts at the origin, i.e., $X_0 = 0$.

4. THE ENVIRONMENT PROCESS

The environment process (EP) is the configuration of traps as seen from the position of an independent random walk $\{X_t : t \geqslant 0\}$. More precisely, it is the process

$$\{\tau_{X_t} \eta_t : t \geqslant 0\} \tag{4.1}$$

defined on the probability space

$$(\Omega^T \times \Omega^{\mathrm{RW}}, \mathscr{A}^T \otimes \mathscr{A}^{\mathrm{RW}}, \mathbb{P}_\eta^T \otimes \mathbb{P}_0^{\mathrm{RW}}) \tag{4.2}$$

i.e., walk and environment of moving traps are *independent*.

In the following proposition we identify the generator of the EP.

Proposition 4.1. $\{\tau_{X_t} \eta_t : t \geqslant 0\}$ is a Markov process with generator L given by

$$Lf(\eta) = L_0 f(\eta) + L_1 f(\eta) \tag{4.3}$$

where L_0 is defined in (2.5) and

$$L_1 f(\eta) = \sum_y p(0, y)[f(\tau_y \eta) - f(\eta)]$$
$$= [L_{\mathrm{RW}} \tau_{(\cdot)} f(\eta)](0) \tag{4.4}$$

Proof. The Markov property of $\{\tau_{X_t} \eta_t : t \geqslant 0\}$ is evident, so we concentrate on proving (4.3) and (4.4). Let $\tilde{\mathbb{E}}_\eta$ be expectation w.r.t. the process $\{\tau_{X_t} \eta_t : t \geqslant 0\}$ starting at $\eta \in E$. Then we can write

$$\begin{aligned}
\frac{\tilde{\mathbb{E}}_n f(\eta_t) - f(\eta)}{t} &= \frac{[\int d(\mathbb{P}_\eta^T \otimes \mathbb{P}_0^{\mathrm{RW}}) f(\tau_{X_t}\eta_t)] - f(\eta)}{t} \\
&= \sum_x \frac{p_t(0, x)}{t} [\mathbb{E}_\eta^T f(\tau_x \eta_t) - f(\eta)] \\
&= \sum_x p_t(0, x) \left(\frac{\mathbb{E}_\eta^T \tau_x f(\eta_t) - \tau_x f(\eta)}{t} \right) \\
&\quad + \frac{\sum_x p_t(0, x)[\tau_x f(\eta) - \tau_0 f(\eta)]}{t}
\end{aligned} \tag{4.5}$$

Taking the limit $t \downarrow 0$ in the r.h.s. of (4.5) yields

$$\begin{aligned}
Lf(\eta) &= \sum_x \delta_{0,x}(L_0 \tau_x f)(\eta) + [L_{\mathrm{RW}} \tau_{(\cdot)} f(\eta)](0) \\
&= L_0 f(\eta) + L_1 f(\eta)
\end{aligned} \tag{4.6}$$

This proves the claim. ■

Proposition 4.2. The measure μ_{eq} is reversible and ergodic for the EP.

Proof. For reversibility we have to show that L is self-adjoint on $L^2(\mu_{\mathrm{eq}})$. By translation invariance of μ_{eq} and by symmetry and translation invariance of the probability kernel $p(x, y)$, it is easy to see that L_1 is a self-adjoint bounded operator on $L^2(\mu_{\mathrm{eq}})$. Since L_0 is self-adjoint by reversibility of μ_{eq}, the generator $L = L_0 + L_1$ is self-adjoint on $L^2(\mu_{\mathrm{eq}})$.

In order to prove ergodicity, suppose that $f \in L^2(\mu_{\mathrm{eq}})$ satisfies $Lf = 0$. Then we have to prove that $f = \int f(\eta)\, \mu_{\mathrm{eq}}(d\eta)$ μ_{eq}-a.s. Now, $Lf = 0$ implies, with the help of Proposition 4.1,

$$(f, Lf) = (f, L_0 f) + (f, L_1 f) = 0 \tag{4.7}$$

By nonnegativity of both $(-L_0)$ and $(-L_1)$, this in turn implies

$$(f, L_0 f) = (f, L_1 f) = 0 \tag{4.8}$$

By the spectral theorem for the self-adjoint nonnegative operator $-L_0$, we have

$$(f, -L_0 f) = \int_0^\infty \lambda\, E_{f,f}(d\lambda) \tag{4.9}$$

where $E_{f,f}(d\lambda)$ is the spectral measure of $(-L_0)$ associated to f. Combining (4.8) and (4.9), we conclude

$$\lambda = 0 \quad E_{f,f}(d\lambda)\text{-a.s.} \tag{4.10}$$

Therefore

$$\int_0^\infty \lambda^2 E_{f,f}(d\lambda) = (L_0 f, L_0 f) = 0 \tag{4.11}$$

and hence $L_0 f(\eta) = 0$, μ_{eq}-a.s. Because μ_{eq} is ergodic for the process of moving traps $\{\eta_t : t \geqslant 0\}$, the latter in turn implies $f = \int f(\eta)\, \mu_{\text{eq}}(d\eta)$ μ_{eq}-a.s. ■

5. THE TRAPPING MODEL

We introduce a *killing function*

$$\Phi: \quad E \to [0, \infty) \tag{5.1}$$

and assume that Φ is continuous and such that

$$\rho = \int \mu_{\text{eq}}(d\eta)\, \Phi(\eta) \in (0, \infty) \tag{5.2}$$

Given Φ, given a configuration of traps $\eta \in E$, and given $x \in \mathbb{Z}^d$, we say that *x is a trapping point of η* iff

$$\Phi(\tau_x \eta) > 0 \tag{5.3}$$

Given the walk $\{X_t : t \geqslant 0\}$ $(X_0 = 0)$ defined in Section 3 and the process of moving traps $\{\eta_t : t \geqslant 0\}$ defined in Section 2, we say that the walker is *killed* at time t when X_t is a trapping point of η_t, i.e., when

$$\Phi(\tau_{X_t} \eta_t) > 0 \tag{5.4}$$

We can then introduce the *survival time* T of the walker:

$$T = \sup\{t \geqslant 0 \mid \text{the walker is not killed at time } s,\ \forall s \in [0, t)\} \tag{5.5}$$

6. LARGE-DEVIATION CONDITIONS ON THE PROCESS OF MOVING TRAPS

In order to prove an exponential estimate for the survival time T, we impose some large-deviation conditions on the process of moving traps. In Section 8 we shall give some examples where these conditions are satisfied.

Define the empirical distribution at time $t>0$ by

$$\mathscr{L}_t=\frac{1}{t}\int_0^t ds\, \delta_{\eta_s}\in\mathscr{P}(E) \tag{6.1}$$

By the ergodic theorem

$$\lim_{t\to\infty} \mathscr{L}_t=\mu_{\text{eq}} \quad \mathbb{P}^T_{\mu_{\text{eq}}}\text{-a.s.} \tag{6.2}$$

Assumption 3. We impose the *large-deviation principle* (LDP) for $\{\mathscr{L}_t: t>0\}$, i.e., $\exists I_2: \mathscr{P}(E)\to\mathbb{R}_+$ lower semicontinuous such that:

(1) $\forall F\subset\mathscr{P}(E)$ closed

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}^T_{\mu_{\text{eq}}}(\mathscr{L}_t\in F)\leqslant -\inf_{\mu\in F} I_2(\mu) \tag{6.3}$$

(2) $\forall G\subset\mathscr{P}(E)$ open

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}^T_{\mu_{\text{eq}}}(\mathscr{L}_t\in G)\geqslant -\inf_{\mu\in G} I_2(\mu) \tag{6.4}$$

Moreover, we suppose that the entropy function I_2 is the lower semicontinuous modification of the Dirichlet form, i.e.,

$$I_2(\mu)=\lim_{\varepsilon\to 0}\inf_{\mu'\in B(\mu,\varepsilon)}\tilde{I}_2(\mu') \tag{6.5}$$

where $B(\mu,\varepsilon)$ is the weak ball of radius ε and centered at μ, and

$$\tilde{I}_2(\mu)=\left(\left(\frac{d\mu}{d\mu_{\text{eq}}}\right)^{1/2}, (-L_0)\left(\frac{d\mu}{d\mu_{\text{eq}}}\right)^{1/2}\right) \tag{6.6}$$

This condition on the entropy function is natural in the reversible context (ref. 3, Section 5.3). If the r.h.s. of (6.6) is not defined, then we put $\tilde{I}_2(\mu)=\infty$.

By (6.3), (6.4), and the contraction principle,[5] the collection

$$\left\{\xi_t=\frac{1}{t}\int_0^t \Phi(\eta_s)\,ds: t>0\right\}$$

satisfies the LDP with entropy function

$$I_1(x)=\inf\left\{I_2(\mu)\,\middle|\,\mu\in\mathscr{P}(E)\text{ and }\int\Phi(\eta)\,\mu(d\eta)=x\right\} \tag{6.7}$$

By (5.2) and the ergodic theorem, $I_1(\rho)=0$. We finally impose the following *nondegeneracy condition* on I_1:

Assumption 4:

$$I_1(x)=0 \Leftrightarrow x=\rho \tag{6.8}$$

7. EXPONENTIAL ESTIMATE FOR THE SURVIVAL TIME

Our main theorem reads:

Theorem 7.1. If $\{\eta_t : t \geqslant 0\}$ satisfies the large-deviation conditions of Section 6, then

$$\limsup_{t\to\infty} \frac{1}{t} \log \tilde{\mathbb{P}}_{\mu_{\text{eq}}}(T \geqslant t) < 0 \tag{7.1}$$

Proof. Abbreviate $\mathbb{P}_{\mu_{\text{eq}}} := \mathbb{P}$:

$$\begin{aligned} \mathbb{P}(T \geqslant t) &= \mathbb{P}(\Phi(\tau_{X_s}\eta_s)=0\ \forall s\in[0,t)) \\ &= \mathbb{P}\left(\int_0^t \Phi(\tau_{X_s}\eta_s)\,ds = 0\right) \\ &= \mathbb{P}\left(\exp\left[-\int_0^t \Phi(\tau_{X_s}\eta_s)\,ds\right] \geqslant 1\right) \\ &\leqslant \mathbb{E}\exp\left[-\int_0^t \Phi(\tau_{X_s}\eta_s)\,ds\right] \end{aligned} \tag{7.2}$$

where the first step follows from the nonnegativity of the killing function Φ and the last step from the Markov inequality. By Proposition 4.2, the generator of the EP is self-adjoint on $L^2(\mu_{\text{eq}})$. Therefore, by the spectral theorem and the Feynman–Kac formula, we have

$$\frac{1}{t}\log \mathbb{E}\exp\left[-\int_0^t \Phi(\tau_{X_s}\eta_s)\,ds\right] \leqslant \Lambda \tag{7.3}$$

where Λ is the greatest eigenvalue of the self-adjoint operator $-\Phi + L_0 + L_1$. By using the variational formula for the greatest eigenvalue of a self-adjoint operator, we obtain

$$\Lambda = \sup_{g: \int g^2 \, d\mu_{eq} = 1} \left[\int -\Phi(\eta) \, g^2(\eta) \, \mu_{eq}(d\eta) + (g, L_0 g) + (g, L_1 g) \right]$$

$$\leqslant \sup_{\mu \in \mathscr{P}(E), \, \mu \ll \mu_{eq}} \left[-\int \Phi(\eta) \, \mu(d\eta) + \left(\left(\frac{d\mu}{d\mu_{eq}} \right)^{1/2}, L_0 \left(\frac{d\mu}{d\mu_{eq}} \right)^{1/2} \right) \right.$$

$$\left. + \left(\left(\frac{d\mu}{d\mu_{eq}} \right)^{1/2}, L_1 \left(\frac{d\mu}{d\mu_{eq}} \right)^{1/2} \right) \right]$$

$$\leqslant \sup_{\mu \in \mathscr{P}(E)} \left[-\int \Phi(\eta) \, \mu(d\eta) - I_2(\mu) \right] \tag{7.4}$$

Here the last step follows from (6.5), (6.6), and the fact that $-L_1$ is nonnegative on $L^2(\mu_{eq})$.

At this stage the random walk is totally out, i.e., the RHS of (7.4) is a function of the process of moving traps only. By Varadhan's formula

$$\sup_{\mu \in \mathscr{P}(E)} \left[-\int \Phi(\eta) \, \mu(d\eta) - I_2(\mu) \right]$$

$$= \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}^T_{\mu_{eq}} \exp \left[-\int_0^t \Phi(\eta_s) \, ds \right] \tag{7.5}$$

By (6.6) and Varadhan's formula

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}^T_{\mu_{eq}} \exp \left[-\int_0^t \Phi(\eta_s) \, ds \right]$$

$$= \lim \frac{1}{t} \log \mathbb{E}^T_{\mu_{eq}} e^{-t\xi_t}$$

$$= \sup_{x \in \mathbb{R}_+} [-x - I_1(x)] \tag{7.6}$$

Because $\rho > 0$, it follows from the lower semicontinuity of I_1 and from the nondegeneracy condition (6.8) that

$$\sup_{x \in \mathbb{R}_+} [-x - I_1(x)] < 0 \tag{7.7}$$

Hence we have proved Theorem 1. ■

8. EXAMPLES

8.1. A Random Walker in a Simple Symmetric Exclusion Process of Traps

In this example the process of moving traps $\{\eta_t : t \geqslant 0\}$ is the simple symmetric exclusion process (SSE). This means

$$E = \{0, 1\}^{\mathbb{Z}^d} \tag{8.1}$$

with the interpretation $\eta(x)=1$ when there is a trap at x, $\eta(x)=0$ otherwise. The trapping points are the points for which $\eta(x)=1$, and therefore

$$\Phi(\eta)=\eta(0) \tag{8.2}$$

The generator of SSE is given by

$$L_0 f(\eta)=\sum_{x\in\mathbb{Z}^d}\sum_{e\in\mathbb{Z}^d:\,|e|=1}\frac{1}{2d}\left[f(\eta^{x,x+e})-f(\eta)\right] \tag{8.3}$$

where $\eta^{x,x+e}$ is obtained from η by exchanging the occupations at lattice points x and $x+e$, i.e.,

$$\eta^{x,x+e}(y)=\eta(y)(1-\delta_{y,x}-\delta_{y,x+e})+\eta(x)\,\delta_{y,x+e}+\eta(x+e)\,\delta_{y,x} \tag{8.4}$$

For the existence and the ergodic theorems of this process we refer to ref. 7, Chapter VIII, Section 1. The reversible and ergodic measures for SSE are the Bernoulli measures $\{\nu_\rho: \rho\in[0,1]\}$ defined by:

(1) $\int \nu(d\eta)\,\eta(0)=\rho$.

(2) Under ν_ρ, $\{\eta(x): x\in\mathbb{Z}^d\}$ is an i.i.d. random field.

Therefore we can pick

$$\mu_{\text{eq}}=\nu_\rho \qquad \text{with} \quad 0<\rho<1 \tag{8.5}$$

The large-deviation conditions of Section 6 are proved for SSE by Landim[6] for $d\geqslant 3$. We thus obtain:

Corollary 8.1. Let T be the survival time of a random walker in an environment of traps that perform an independent SSE on $\mathbb{Z}^d$ with $d\geqslant 3$, starting from ν_ρ ($\rho\in(0,1)$). Then

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(T\geqslant t)<0 \tag{8.6}$$

8.2. A Random Walker in a Simple Zero-Range Process of Traps

In this example the process $\{\eta_t: t\geqslant 0\}$ is the simple zero-range process (SZRP). This is an infinite system of independent simple random walkers. In this case

$$E=\mathbb{N}^{\mathbb{Z}_d} \tag{8.7}$$

where $\eta(x)$ represents the number of traps at lattice site x. The traps are the points where $\eta(x) \neq 0$, i.e., the killing function can be taken

$$\Phi(\eta) = \eta(0) \tag{8.8}$$

The generator of SZRP is given by

$$L_0 f(\eta) = \sum_{x \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d:\, |e| = 1} \eta(x) \frac{1}{2d} [f(\eta^{x,x+e}) - f(\eta)] \tag{8.9}$$

where $\eta^{x,x+e}$ is obtained from η by removing a trap at lattice site x and putting it at $x+e$, i.e.,

$$\eta^{x,x+e}(y) = \eta(y) - \delta_{y,x} + \delta_{y,x+e} \tag{8.10}$$

The reversible and ergodic measures for SZRP are the Poisson measures $\{\mu_\rho : \rho \in (0, 8)\}$ defined by:

(1) $\mu_\rho(\eta(x) = n) = (\rho^n/n!)\, e^{-\rho}$.

(2) Under μ_ρ, $\{\eta(x) : x \in \mathbb{Z}^d\}$ is an i.i.d. field.

Therefore, we can pick

$$\mu_{\text{eq}} = \mu_\rho \qquad \text{with} \quad \rho > 0 \tag{8.11}$$

The large-deviation properties of Section 6 are easily verified in dimension $d \geq 3$ with the help of ref. 2. Therefore we obtain:

Corollary 8.2. Let T be the survival time of a random walker in an environment of traps that perform an independent SZRP starting from μ_ρ, $\rho > 0$. Then in $d \geq 3$

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}(T \geq t) < 0 \tag{8.12}$$

9. A RANDOM WALKER IN A SIMPLE ZERO-RANGE PROCESS OF TRAPS: LOWER BOUND

In the general context of Theorem 7.1 we did not succeed in proving an exponential lower bound. In the case of SZRP discussed in Section 8.2 we can derive an explicit formula for $P(T \geq t)$ containing the range of a random walk (Lemma 9.1 below). From this formula we can derive a lower bound and we can deal with the lower-dimensional case (i.e., $d \leq 2$) in which $P(T \geqslant t)$ decays subexponentially. First we introduce some more notation. For $\eta \in \mathbb{N}^{\mathbb{Z}^d}$ the SZRP starting from η is a collection of independent random walks

$$((\{X_i^x(t): t \geqslant 0\})_{i=1}^{\eta(x)})_{x \in \mathbb{Z}^d} \tag{9.1}$$

Given a simple random walk $X = \{X(t): t \geqslant 0\}$ $(X_0 = 0)$ we use the notation P_X for its path space measure and $\mathbb{E}_X$ for the corresponding expectation. For a process $Z = \{Z(t): t \geqslant 0\}$ on $\mathbb{Z}^d$ we define *the range at time t*:

$$R(Z, t) = \sum_x 1_{\{Z(s) = x \text{ for some } s \in [0, t]\}} \tag{9.2}$$

For simple random walk we abbreviate $R(X, t) = R_t$. Finally for two processes X and Y on $\mathbb{Z}^d$ we define $X - Y$ to be the pointwise difference, i.e., $X - Y = \{X(t) - Y(t): t \geqslant 0\}$.

In Lemma 9.1 below we prove that the survival probability $P(T \geqslant t)$ of Corollary 8.2 can be written as a function of the range of the difference of two independent random walks.

Lemma 9.1. Let T be the survival time of a simple random walker in an environement of traps that perform an independent SZRP on $\mathbb{Z}^d$ starting from μ_ρ. Then

$$P(T \geqslant t) = \mathbb{E}_X[\exp[-\rho \mathbb{E}_Y R(X - Y, t)]] \tag{9.3}$$

Proof.

$$\begin{aligned}
P(T \geqslant t) &= \int \mu_\rho(d\eta)\, P\,(X_i^x(s) - X(s) \neq 0\ \forall i = 1,..., \eta(x), \forall x, \forall s \in [0, t)) \\
&= \int dP_X \int \mu_\rho(d\eta) \prod_x \prod_{i=1}^{\eta(x)} \int dP_{X_i} 1_{\{X_i(s) \neq X(s) + x\ \forall s \in [0, t)\}} \\
&= \int dP_X \int \mu_\rho(d\eta) \prod_x \left(\int dP_Y 1_{\{Y(s) \neq X(s) + x\ \forall s \in [0, t)\}} \right)^{\eta(x)} \\
&= \int dP_X \prod_x [\exp(-\rho) \exp(\rho P_Y \{Y(s) \neq X(s) + x\ \forall s \in [0, t)\})] \\
&= \int dP_X \exp\left[-\rho \int dP_Y \sum_x 1_{\{Y(s) - X(s) = x\ \exists s \in [0, t)\}} \right] \\
&= \int dP_X \exp\left[-\rho \int dP_Y\, R(Y - X, t \right]
\end{aligned} \tag{9.4}$$

Here in the second step we used the independence of the walkers in the SZRP and in the fifth step we used the Poisson caracter of μ_ρ. ■

Theorem 9.2. In the context of Lemma 8.3:

$$P(T \geqslant t) \geqslant \exp(-\rho \mathbb{E} R_{2t}) \tag{9.5}$$

Proof. From (9.3) and Jensen's inequality we conclude

$$P(T \geqslant t) \geqslant \exp[-\rho \mathbb{E}_X \mathbb{E}_Y R(X-Y, t)] \tag{9.6}$$

Since the difference of two independent simple random walks is a simple random walk at twice the speed, we obtain

$$\mathbb{E}_X \mathbb{E}_Y R(X-Y, t) = \mathbb{E} R_{2t} \tag{9.7}$$

Combination of (9.6) and (9.7) yields the claim. ■

From the theory of simple random walk we know that[8]

$$\begin{aligned} \mathbb{E} R_{2t} &\sim (2t)^{1/2} \quad (t \to \infty) \quad \text{in } d=1 \\ &\sim \frac{2\pi t}{\log(2t)} \quad (t \to \infty) \quad \text{in } d=2 \\ &\sim 2\gamma_d t \quad (t \to \infty) \quad \text{in } d \geqslant 3 \end{aligned} \tag{9.8}$$

where γ_d is the probability of never returning to the origin.

Therefore, Theorem 9.2 yields a subexponential decay of $P(T \geqslant t)$ in dimension $d \geqslant 2$, and an exponential lower bound in $d \geqslant 3$.

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